Welcome to

FUNDAMENTAL MATHEMATICS

FOR INDUSTRY EDUCATION

Lesson 3 Part 2

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Inverse Matrix

Definition

Let A be an n × n square matrix .if there exists an n × n matrix B such that

$$AB = BA = I_n$$

Where I_n is the identity matrix of order n then B is called the inverse matrix of A. The inverse of A is denoted by

$$A^{-1}$$

Inverse Matrix

Existence Condition

A square matrix A has an inverse if and only if

$$\det(A) \neq 0$$

If $\det(A) = 0$, matrix A has no inverse.

Basic Properties

If A is invertible, then:

- The inverse of A is unique
- $(A^{-1})^{-1} = A$
- $AA^{-1} = A^{-1}A = I_n$

Inverse Matrix

Inverse of a 2×2 Matrix

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if det (A) = $ad - cb \neq 0$

then the inverse of A is given by

$$\boxed{A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Inverse Matrix

General Formula for an $n\times n$ Matrix (Adjugate Matrix Method)

Let

A be an $n\times n$ square matrix with $\det(A)\neq 0$. The inverse of A is defined by

$$\boxed{A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)}$$

Inverse Matrix

General Formula for an n × n Matrix (Adjugate Matrix Method)

When

$$\operatorname{adj}(A) = \left(\operatorname{cof}(A)\right)^T$$

And the cofactor matrix is $\operatorname{cof}(A) = \left[C_{ij}\right] \Rightarrow C_{ij} = (-1)^{i+j} M_{ij}$

with M_{ij} denoting the minor of the element a_{ij}

Inverse Matrix

Minor

Definition

Let $A = [a_{ij}]$ be an $n \times n$ square matrix.

The **minor of the element** a_{ij} ,denoted by

$$M_{ij}$$

Determinant of the submatrix obtained by:

deleting the i row

deleting the j column

from the matrix A

Example

Given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find M_{11}

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find M_{12}

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find M_{13}

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Example

Given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find M_{21}

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find M_{22}

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find M_{23}

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Example

Given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find M_{31}

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find M_{32}

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find M_{33}

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Example

Given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Inverse Matrix

Cofactor (cof)

Definition

Let $A = [a_{ij}]$ be an $n \times n$ square matrix.

The **cofactor of the element** a_{ij} ,denoted by C_{ij} ,is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Where M_{ij} is the **minor** of a_{ij} the determinant of the submatrix obtained by deleting the i row and j column of A.

Inverse Matrix

Cofactor matrix 2×2

When $M_{ij} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$

From $C_{ij} = (-1)^{i+j} M_{ij}$

C_{11} is $(-1)^{1+1} M_{11} = (-1)^2 M_{11} \Rightarrow M_{11}$

C_{12} is $(-1)^{1+2} M_{12} = (-1)^3 M_{12} \Rightarrow -M_{12}$

C_{21} is $(-1)^{2+1} M_{21} = (-1)^3 M_{21} \Rightarrow -M_{21}$

C_{22} is $(-1)^{2+2} M_{22} = (-1)^4 M_{22} \Rightarrow M_{22}$

$$\boxed{\left[C_{ij}\right] = \begin{bmatrix} M_{11} & -M_{12} \\ -M_{21} & M_{22} \end{bmatrix}}$$

Inverse Matrix

Cofactor matrix 3×3

When $M_{ij} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$

From $C_{ij} = (-1)^{i+j} M_{ij}$

C_{11} is $(-1)^{1+1} M_{11} = (-1)^2 M_{11} \Rightarrow M_{11}$

C_{12} is $(-1)^{1+2} M_{12} = (-1)^3 M_{12} \Rightarrow -M_{12}$

C_{13} is $(-1)^{1+3} M_{13} = (-1)^4 M_{13} \Rightarrow M_{13}$

C_{21} is $(-1)^{2+1} M_{21} = (-1)^3 M_{21} \Rightarrow -M_{21}$

C_{22} is $(-1)^{2+2} M_{22} = (-1)^1 M_{22} \Rightarrow M_{22}$

C_{23} is $(-1)^{2+3} M_{23} = (-1)^5 M_{23} \Rightarrow -M_{23}$

C_{31} is $(-1)^{3+1} M_{31} = (-1)^4 M_{31} \Rightarrow M_{31}$

C_{32} is $(-1)^{3+2} M_{32} = (-1)^5 M_{32} \Rightarrow -M_{32}$

C_{33} is $(-1)^{3+3} M_{33} = (-1)^6 M_{33} \Rightarrow M_{33}$

$$\boxed{\left[C_{ij}\right] = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}}$$

Inverse Matrix

Cofactor matrix 2×2

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

Cofactor matrix 3×3

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Cofactor matrix 4×4

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Inverse Matrix

Cofactor matrix 5×5

$$\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

Cofactor matrix 6×6

$$\begin{bmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \end{bmatrix}$$

Example

Find cofactor of matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow M_{ij} = \begin{bmatrix} -3 & -6 & -3 \\ -6 & -12 & -6 \\ -3 & -6 & -3 \end{bmatrix}$$

Example

Given

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Find A^{-1}

Example

Given

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Find M_{ij}

Find M_{11}

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Find M_{12}

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Find M_{13}

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Example

Given

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Find M_{ij}

Find M_{21}

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Find M_{22}

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Find M_{23}

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Example

Given

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Find M_{ij}

Find M_{31}

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Find M_{32}

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Find M_{33}

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Example

Given

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Example

Given

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Inverse Matrix

General Formula for an n × n Matrix (Adjugate Matrix Method)

Procedure

Step 1: Compute the determinant of A

Step 2: Compute the Minor Matrix $\rightarrow M = [M_{ij}]$

Step 3: Compute the Cofactor Matrix $\rightarrow C = [C_{ij}] \rightarrow C_{ij} = (-1)^{i+j}M_{ij}$

Step 4: Compute the Adjugate Matrix $\rightarrow \text{adj}(A) = (\text{cof}(A))^T$

Step 5: Compute the Inverse Matrix $\rightarrow A^{-1} = \dfrac{1}{\det(A)}\text{adj}(A)$