มหาวิทยาลัยราชภัฏนครปฐม
NAKHON PATHOM RAJABHAT UNIVERSITY

Welcome to

FUNDAMENTAL MATHEMATICS

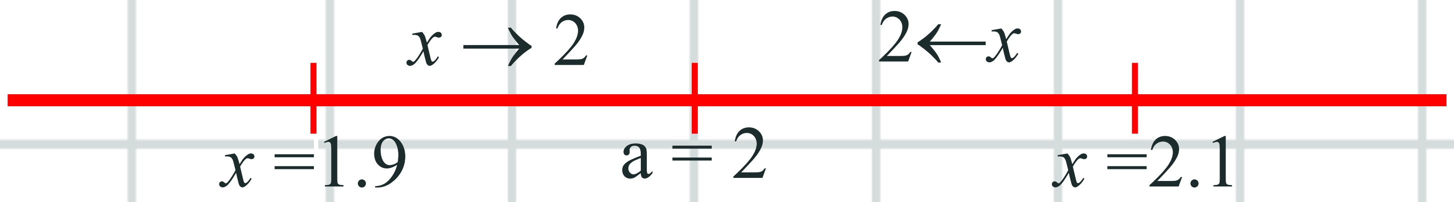
FOR INDUSTRY EDUCATION

Lesson 04

Limit and Continuity

KHACHONSAK SIMEK

9641103 Fundamental Mathematics for Industry Education



Limit and Continuity of a Function

Finding the value of a function $f(x)$ as x approaches a means determining the value that the function approaches when x gets close to a, but x is not equal to a.

$$f(x) = x + 2$$

$x < 2$	
x	$f(x) = x + 2$
1	
1.5	
1.9	
1.99	
1.999	
1.9999	
$\vdots$	

$$\lim_{x \to 2^-} (x + 2) = 4$$

$$\lim_{x \to 2^+} (x + 2) = 4$$

$$\lim_{x \to 2} (x + 2) = 4$$

$x > 2$	
x	$f(x) = x + 2$
3	
2.5	
2.1	
2.01	
2.001	
2.0001	
$\vdots$	

Limit and Continuity of a Function

Let $y = f(x)$ be any function, and let *a, L, A,* and *B* be real numbers. The following symbols are used to represent meanings related to limits:

$\lim\limits_{x\to a} f(x) = L$ means that the limit of the function *f* as *x* approaches *a* is equal to *L*.

$\lim\limits_{x\to a^-} f(x) = A$ means that the limit of the function *f* as *x* approaches *a* from the left is equal to *A*. This is called **the left-hand limit of *f*.**

$\lim\limits_{x\to a^+} f(x) = B$ means that the limit of the function *f* as *x* approaches *a* from the right is equal to *B*. This is called **the right-hand limit of *f*.**

Limit and Continuity of a Function

There for $\boxed{\lim_{x\to a} f(x) = L \text{ when } \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)}$

A function f is **continuous** at $x = a$ in the domain of f if and only if the following conditions are satisfied:

Can find the $\lim_{x\to a} f(x)$

Can find the $f(a)$

$$\lim_{x\to a} f(x) = f(a)$$

If any one of these conditions is not satisfied, then the function f is said to be discontinuous at $x = a$.

Example

Construct a table showing the values of the given function at points near a (but not equal to a) for each of the following functions, in order to determine the left-hand limit, the right-hand limit, and the limit of the function (if it exists).

$f(x) = 3x^2 - 2x$ when $a = 2$

$x < 2$	
x	$f(x) = 3x^2 - 2x$

$x > 2$	
x	$f(x) = 3x^2 - 2x$

Example

Construct a table showing the values of the given function at points near a (but not equal to a) for each of the following functions, in order to determine the left-hand limit, the right-hand limit, and the limit of the function (if it exists).

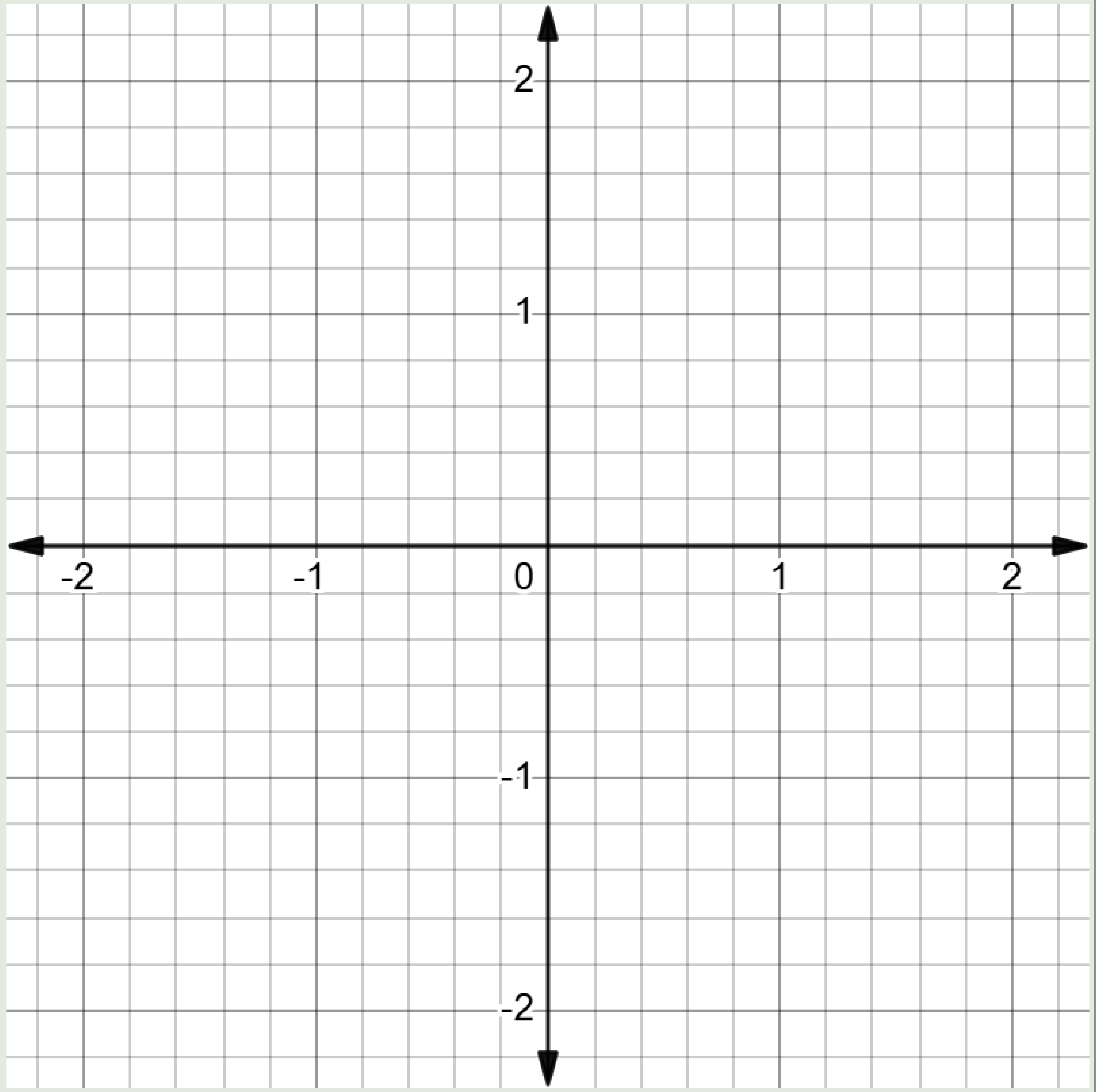
$f(x)=\sqrt{x+1}$ when $a=8$

$\boldsymbol{x<8}$	
x	$f(x)=\sqrt{x+1}$

$\boldsymbol{x>8}$	
x	$f(x)=\sqrt{x+1}$

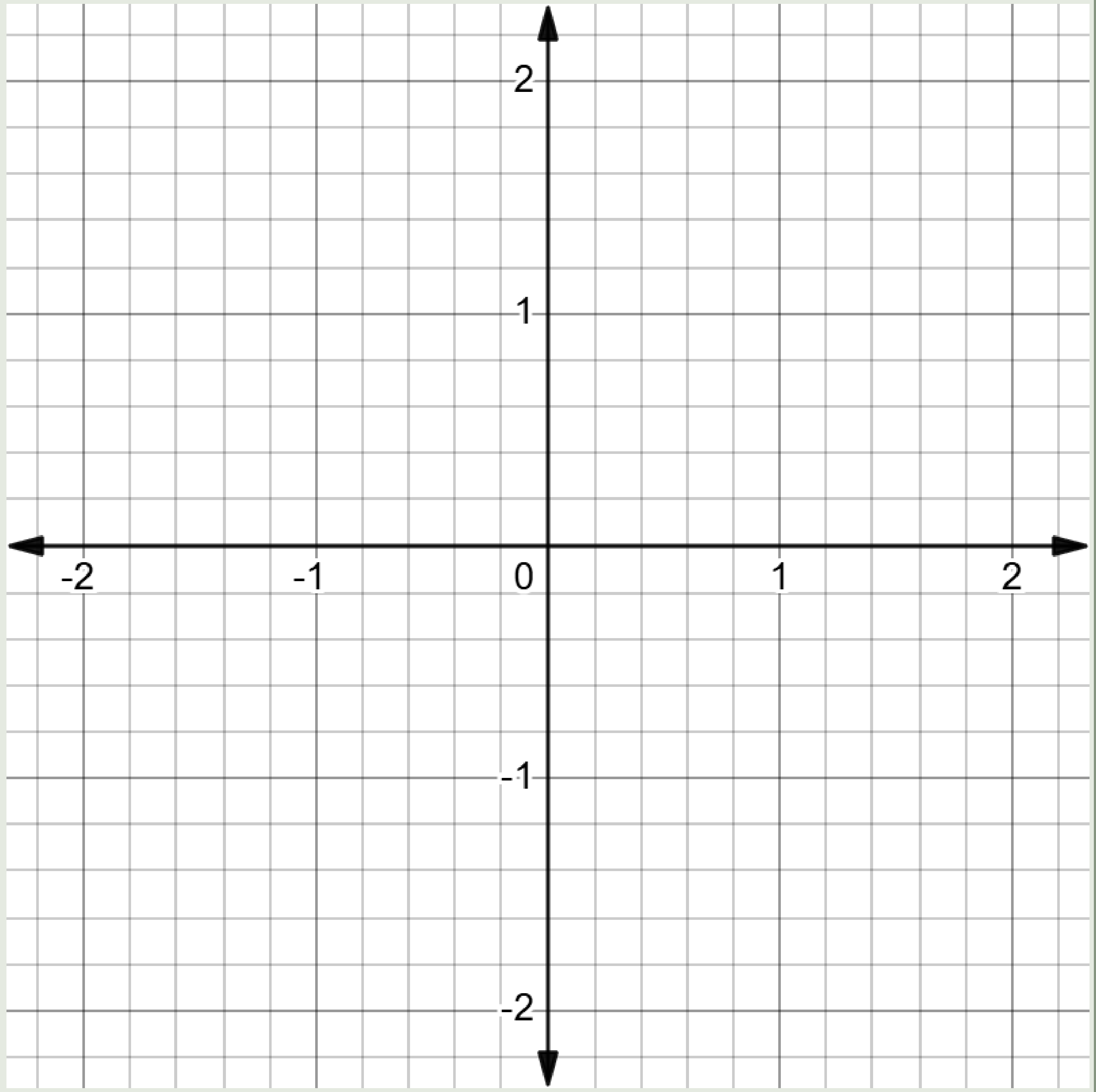
Example

Consider the graph of the given function. Analyze the left-hand limit, the right-hand limit, and the limit of the function (if it exists). Then discuss whether the function is continuous at $x = a$, providing a clear justification.



Example

Consider the graph of the given function. Analyze the left-hand limit, the right-hand limit, and the limit of the function (if it exists). Then discuss whether the function is continuous at $x = a$, providing a clear justification.



Evaluating Limits

1. $\lim_{x\to a} C = C$ when $f(x) = C$ is constant function
2. $\lim_{x\to a} f(x) = a$
3. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then

    3.1 $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$

    3.2 $\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M$

    3.3 $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}, M \neq 0$

Evaluating Limits

4. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$, when $f(x) = L$

and $L \geq 0, \sqrt[n]{L}$ The value of L exists when n is a positive integer.

$L < 0, \sqrt[n]{L}$ The value of L exists when n is a negative integer.

*This rule can also be applied to the left-hand limit and the right-hand limit.

Example Evaluate the limit.

$$\lim_{x \to 2}(x^2 + 7x - 3)$$

Example Evaluate the limit.

$$\lim_{x\to 2}(x^2+2)(x^3+4x^2-1)$$

Example Evaluate the limit.

$$\lim_{x \to 1^+} \frac{x^2+1}{\sqrt{x+1}}$$

Continuity of a Function

A function f is continuous at $x = a$ if and only if the following conditions are satisfied.

1. Can find the $\lim_{x \to a} f(x)$
2. Can find the $f(a)$
3. $\lim_{x \to a} f(x) = f(a)$

Left continuity and right continuity

A function f is left-continuous at $x = a$

if $\lim_{x \to a^-} f(x) = f(a)$

A function f is right-continuous at $x = a$

If $\lim_{x \to a^+} f(x) = f(a)$

A function f is continuous on the closed interval $[a,b]$ if and only if the following conditions are satisfied.

1. f is continuous on the open interval (a,b)
2. f is left-continuous at $x = b$
3. f is right-continuous at $x = a$

Example Given a function $f(x)$, determine whether f is continuous at $x = a$, and explain why.

$f(x) = 3x - 4, \; a = 2$

Consider $x = 2$

1. $f(2) = 3(2) - 4 = 2$

2. $\lim_{x \to 2} f(x) = \lim_{x \to 2}(3x - 4) = 3(2) - 4 = 2$

3. $f(2) = \lim_{x \to 2}(3x - 4) = 2$

Therefor f is continuous at $x = 2$ because it satisfies all three conditions.

Example Given a function $f(x)$, determine whether f is continuous at $x = a$, and explain why.

$$f(x) = \frac{1}{x+2}, \ a = -2$$

Consider x = -2

1. $f(-2) = \dfrac{1}{(-2)+2}$ function is not defined at $x = -2$ because the denominator equals zero.

Therefor f is continuous at $x = -2$ because function is not defined.

Example Given a function $f(x)$, determine whether f is continuous at $x = a$, and explain why.

$$f(x) = \begin{cases} x & \text{when } x < -2, \\ 2x & \text{when } x \geq -2 \end{cases} \qquad a = -2$$

Consider $x = -2$

1. $f(-2) = 2(-2) = -4$

2. $\lim\limits_{x \to -2^-} (x) = \lim\limits_{x \to -2^-} (-2) = -2$

   $\lim\limits_{x \to -2^+} (2x) = \lim\limits_{x \to -2^+} (2(-2)) = -4$

$\lim\limits_{x \to -2^-} f(x) \neq \lim\limits_{x \to -2^+} f(x) \to \lim\limits_{x \to -2} f(x)$ is not define

Therefor f is continuous at $x = -2$ because $\lim\limits_{x \to -2} f(x)$is not defined.

Limits as x increases or decreases without bound

$$\lim_{x\to+\infty}\frac{1}{x^n}=0$$

$$\lim_{x\to-\infty}\frac{1}{x^n}=0$$

if n is a positive integer

If direct substitution gives $\frac{\infty}{\infty}$, the limit cannot be determined immediately.

In this case, divide both the numerator and the denominator by the highest power of the variable.

Example Evaluate the limit.

$$\lim_{x \to +\infty} \frac{-x^4 + 2x^2 + 3}{2x^4}$$

Example

Evaluate the limit.

$$\lim_{x \to -\infty} \frac{x^3 - 37x^2 + 13^{91}}{2x^3 - 142}$$

Exercise

1. Construct a table showing the values of the given function at points near a (but not equal to a) for each of the following functions, in order to determine the left-hand limit, the right-hand limit, and the limit of the function (if it exists).

1.1 $f(x) = \dfrac{x^3 - 4x}{x^2 + 4x}, a = 0$

1.2 $f(x) = \dfrac{\sqrt{x} - 2}{x - 4}, a = 4$

Exercise

2. Consider the graph of the given function. Analyze the left-hand limit, the right-hand limit, and the limit of the function (if it exists). Then discuss whether the function is continuous at x = a, providing a clear justification.

2.1 $$f(x) = \begin{cases} \dfrac{(x^2-1)}{(x+1)}, & x \neq -1,\ a = -1 \\ 1, & x = -1 \end{cases}$$

2.2 $$f(x) = \begin{cases} \dfrac{x2-x-6}{x-3}, & x \neq -3,\ a = 2,3 \\ 5, & x = 3 \end{cases}$$

Exercise

3. Evaluate the limit

3.1 $\lim\limits_{x\to 4}\left(1-\frac{x^2}{2}\right)\left(1+x^3\right)$

3.2 $\lim\limits_{x\to -3}\frac{x^2+2x}{x^3+1}$

3.3 $\lim\limits_{x\to 4}\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$

3.4 $\lim\limits_{x\to 2}\left(\frac{x^2}{4}-\frac{1}{x}\right)^{-1}$

3.5 $\lim\limits_{x\to 3}\left(1+\frac{1}{x+3}\right)^{-1}$